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Applying Fault-Tolerant Solutions of Circulant Graphs to Multidimensional Meshes

A. A. FARRAG AND S. LOU

Faculty of Computer Science

Dalhousie University

Halifax, N.S., Canada, B3H 3J5

farrag@cs.dal.ca

ShituoLou@msn.com

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Abstract—Recently, circulant graphs have received a lot of attention; and a new method was proposed for designing fault-tolerant solutions for any given circulant graph. This method works by partitioning the offsets of the graph in many ways; each leads to one or more solutions. By comparing all these solutions, we can find the one with the least node-degree. In this paper, we shall first review this method; and then re-examine its applications to the design of k -fault-tolerant meshes (for all possible values of k). Our results demonstrate that the solutions obtained (for both two and three-dimensional meshes) are efficient. © 2005 Elsevier Ltd. All rights reserved.

Keywords—Algorithms, Fault-tolerance, Network architectures, Circulant graphs, Meshes.

1. INTRODUCTION

To speed-up the computation of many applications, a parallel computer employing a large number of processors can be used. The basic idea is to divide the application into smaller tasks that can be distributed, and executed, over a network of processors. The main objective is to reduce the time needed to perform the application by executing as many tasks simultaneously as possible. However, as the number of processors in the network increases, the likelihood of failure also grows. For this reason, fault-tolerance has become a major concern in designing multiprocessor networks [1].

Tolerating failure is usually achieved by introducing redundancy, that is, by adding some extra processors and links to the network. In this case, when some of the processors fail, the system can be reconfigured to bypass the defective components. Extending the network to include fault-tolerance capabilities is known in literature as the fault-tolerant extension problem [2]. This has been previously examined for many architectures, such as rings [3–5], stars [6,7], meshes [3–5,8–15], and hypercubes [5,8–10,15–19].

Recently, the circulant graph configuration has received a lot of attention. It turns out that some important architectures (such as rings and meshes, for examples) can be modeled as cir-

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culant graphs. A new method for constructing k -fault-tolerant solutions for any given circulant graph was proposed in [4]. The basic idea behind this method is to partition the offsets of the graph in many ways; each leads to a different solution. By comparing the family of all solutions generated, we can find the one with the least node-degree. We review this method in this paper, and re-examine its applications to the design of k -fault-tolerant meshes. Our results improve those obtained earlier in [4].

Designing k -fault-tolerant meshes can also be done by the method proposed in [3], which exploits the automorphism properties of circulant graphs. However, their solutions are not efficient in node-degree. Similar upper-bounds to those developed here for two-dimensional meshes were proven independently in [8]. Their solutions (that are based on circulant graphs) are actually special cases of those generated by the formulation (of offsets partitioning) in [4]. Their diagonal-graph method [8] for solving meshes is also efficient in node-degree, and outperforms some of the upper-bounds developed here. However, over some ranges of k , our solutions are actually cheaper. The methods of [9,14] are also very efficient in node degree, but (unlike our method) their solutions require more spare nodes than necessary.

First, we review the formulation proposed in [4] for solving circulant graphs (in Section 3). Then, we re-examine its applications to the design of k -fault-tolerant solutions for both two and three-dimensional meshes (in Sections 4 and 5). We develop new upper-bounds, and construct efficient solutions (for all possible values of k). But first, we present the background material needed below (in Section 2).

2. BACKGROUND

In this section, we develop a formulation for the fault-tolerant extension problem and present the main concepts used throughout the paper.

DEFINITION 2.1. k -FT EXTENSIONS. A graph H is said to be a k -fault-tolerant (or k -ft) extension of a graph G if for every subset S of k nodes in H , the induced subgraph $H-S$ that is obtained by removing all k nodes of S from H must contain a subgraph isomorphic to G .

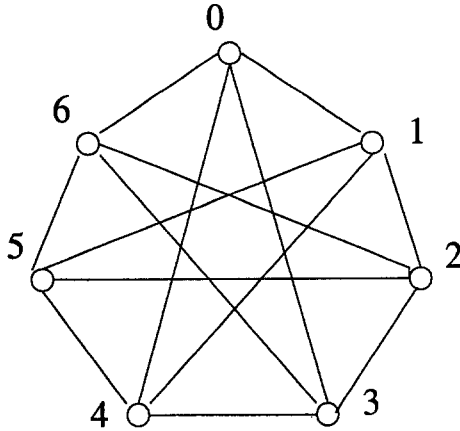
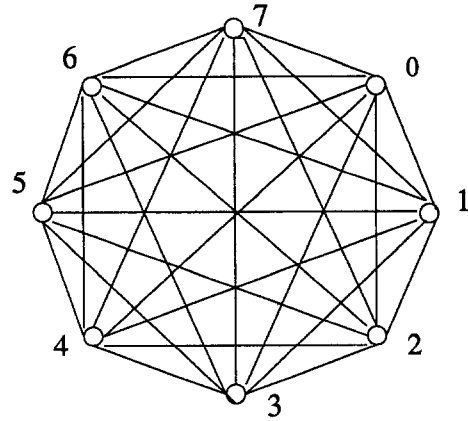
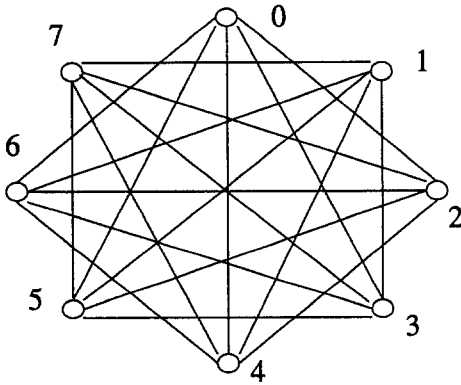
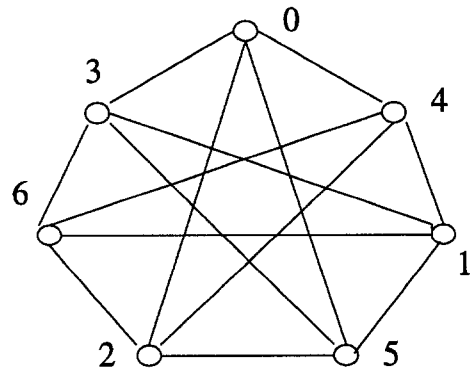
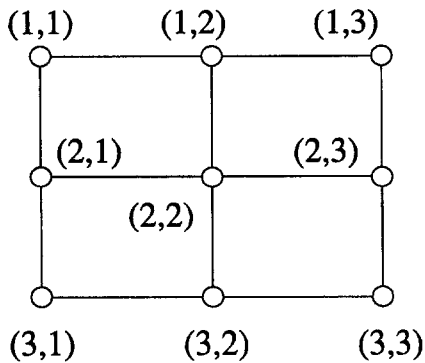
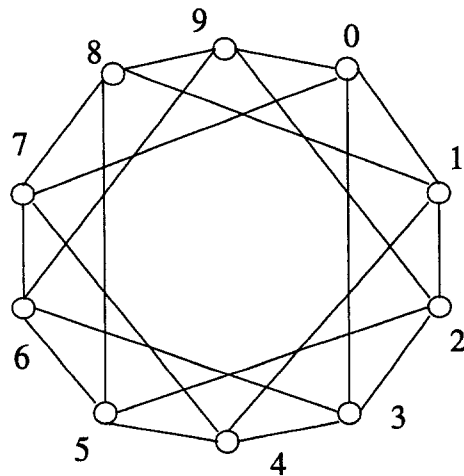
For example, the graph shown below in Figure 1 has a 1-ft solution given in Figure 2 (which is constructed according to the theorem given below). Similarly, the graph in Figure 5 has a 1-ft extension shown in Figure 6 (which is constructed by a method given later in Section 4). In designing a k -ft solution H of a graph G , we use the minimum number of spare nodes, i.e., H exactly k more nodes than G .

DEFINITION 2.2. CIRCULANT GRAPH. An n -node circulant graph is defined by a set of nodes numbered $\{0, 1, \dots, n-1\}$, and a set of integers called offsets, denoted as $A = \{a_1, a_2, \dots, a_i\}$. Two nodes, x and y , are joined by an edge if and only if there is an offset a_i , such that $x - y = \pm a_i \bmod n$. For example, the seven-node circulant graph shown in Figure 1 has the offsets $\{1, 3\}$.

In the above definition, if we replaced an offset a_i by an offset $n - a_i$, (or by $-a_i$), the graph will not change, i.e., it will have the same set of edges. Throughout this paper, we shall use only positive offsets in the range 1 to $\lfloor n/2 \rfloor$ that is, offsets outside this range will be converted into equivalent values within the range.

In what follows, we shall denote an n -node circulant graph G with offsets $\{a_1, a_2, \dots, a_i\}$ as $G[a_1, a_2, \dots, a_i : n]$. (For instance, the graph in Figure 1 is $G[1, 3 : 7]$.) The basic method for constructing a k -ft extension of a circulant graph is based on the following theorem which was proven (by Dutt and Hayes) in [3].

THEOREM 2.1. *Given a graph $G[a_1, a_2, \dots, a_i : n]$, we can construct an $(n + k)$ -node circulant graph H that is a k -ft extension of G where the offsets of H consist of the union $\{a_1, a_1 + 1, a_1 + 2, \dots, a_1 + k\} \cup \{a_2, a_2 + 1, a_2 + 2, \dots, a_2 + k\} \cup \dots \cup \{a_i, a_i + 1, a_i + 2, \dots, a_i + k\}$. ■*

Figure 1. A graph $G[1,3:7]$.Figure 2. A 1-ft of $G[1,3:7]$.Figure 3. A 1-ft of $G[1,3:7]$.Figure 4. A subgraph of Figure 3 that excludes the faulty node 7, and is isomorphic to $G[1,3:7]$.Figure 5. A mesh $M[3,3]$.Figure 6. A 1-ft of $M[3,3]$.

Clearly, a k -ft extension of a graph G , constructed according to the above theorem, will be most efficient if the offsets of G are all consecutive. Otherwise, the solution may not be efficient in node-degree as explained below.

EXAMPLE 2.1. Consider again the graph $G[1,3:7]$ shown in Figure 1, and its 1-ft solution given in Figure 2 whose offsets are $\{1, 2, 3, 4\}$. A cheaper 1-ft solution for G with only three offsets $\{2, 3, 4\}$ can be constructed as shown in Figure 3. To see why, note that removing any node from the solution in Figure 3 will still leave a subgraph that contains a copy isomorphic to G (as explained in Figure 4).

In general, a large family of k -ft solutions can be constructed for any given circulant graph. To compare the cost of these solutions, we count only the number of offsets, since it relates directly to node-degree, i.e., a solution with m offsets has a node-degree equal to $2m$ or $2m - 1$. In counting the offsets, however, we must eliminate duplicates.

3. ALGORITHM FOR CIRCULANT GRAPHS

Many important parallel architectures (such as rings and meshes, for examples) can be modeled as circulant graphs. As a result, these graphs have received a lot of attention, and a new method was recently developed (in [8]) for designing fault-tolerant solutions for them. We review this method in this section.

In the rest of this paper, the following notations is used. The greatest common divisor (\gcd) of two integers x and y will be denoted $\gcd(x, y)$. The inverse of an integer $x \bmod n$ is denoted x^{-1} . Note that (from elementary number theory [20]) we know that $x * x^{-1} \pmod n = 1$. Two integers x and y are called relatively primes if $\gcd(x, y) = 1$.

Given a circulant graph $G[a_1, a_2, \dots, a_i : n]$ and an integer k , the algorithm [4] partitions the offsets of G in many ways; each leads to a different k -ft solution. By comparing the solutions generated, we can select the one with the least node-degree. Partitioning the offsets of G at every step is done by selecting an integer m that is relatively prime to n , that is, $\gcd(n, m) = 1$, as explained below.

DEFINITION 3.1. PARTITIONING SEQUENCES. Let n and m be any pair of integers, such that $\gcd(n, m) = 1$ and $n > m > 0$. We define an ordered sequence, based on n and m , denoted $S(n, m) = \langle s_1, s_2, \dots, s_{\lfloor n/2 \rfloor} \rangle$ where the i^{th} element in this sequence is computed as follows, if $[i * m \pmod n] \leq \lfloor n/2 \rfloor$, then $s_i = [i * m \pmod n]$; otherwise, $s_i = n - [i * m \pmod n]$. For instances, for $n = 7$ and $m = 3$, $S(7, 3) = \langle 3, 1, 2 \rangle$, and for $n = 14$ and $m = 5$, $S(14, 5) = \langle 5, 4, 1, 6, 3, 2, 7 \rangle$.

It is not difficult to show that the sequence $S(n, m)$ contains all integers from 1 to $\lfloor n/2 \rfloor$, that is, it includes the whole range of valid offsets. After generating $S(n, m)$, it will be used to divide the offsets of G as explained below.

DEFINITION 3.2. MAXIMAL m -DISTANCE SUBSET. A subset of a set of offsets of an n -node circulant graph is called an m -distance subset, where m is an integer, such that $\gcd(n, m) = 1$ and $n > m > 0$, if there is a subsequence of consecutive elements inside $S(n, m)$ which contains all elements in this subset. Further, an m -distance subset is called maximal if it is not contained in any other m -distance subset.

DEFINITION 3.3. m -DISTANCE PARTITION. Let $P(A, n, m)$ denote a collection of m -distance subsets defined over a set A of offsets of an n -node circulant graph. Then, $P(A, n, m)$ will be called an m -distance partition of A , if every m -distance subset in $P(A, n, m)$ is maximal and every offset in A appears (in one subset) in $P(A, n, m)$.

EXAMPLE 3.1. Consider a 16-node circulant graph with the offsets $\{1, 2, 4, 6, 7\}$. Since the sequence $S(16, 1) = \langle 1, 2, 3, 4, 5, 6, 7, 8 \rangle$, then a one-distance partition of these offsets consists of the subsets $\{\{1, 2\}, \{4\}, \{6, 7\}\}$. Similarly, since $S(16, 3) = \langle 3, 6, 7, 4, 1, 2, 5, 8 \rangle$, therefore, a three-distance partition of the offsets consists of only one subset $\{\{1, 2, 4, 7, 6\}\}$.

The procedure for partitioning the offsets can now be presented below. (In this procedure, the set A denotes the offsets of the graph G , n denotes the number of nodes; and m is any integer that is relatively primes to n , where $n > m > 0$.)

PROCEDURE. PARTITION (A, n, m) .

- (1) Construct the sequence $S(n, m) = \langle s_1, s_2, \dots, s_{\lfloor n/2 \rfloor} \rangle$ as explained before.
- (2) For every element s_i in $S(n, m)$, if s_i appears as an offset in A , we keep it, otherwise, we replace s_i in $S(n, m)$ by a special separation symbol, say “&”.
- (3) For every maximal subsequence in $S(n, m)$ that does not include the symbol “&”, from an m -distance subset corresponding to this subsequence.
- (4) Return a partition $P(A, n, m)$ consisting of all m -distance subsets formed above.

Since $S(n, m)$ contains $\lfloor n/2 \rfloor$ elements, and each of them can be checked in Step 2 in only $O(\log |A|)$ time, (if A were sorted first), therefore, the cost of the above procedure is $O(n \log |A|)$.

Let G_m denote the graph obtained by partitioning the offsets of G as explained in the above procedure. By repeating this procedure for different values of m , say $m = m_1, m_2, \dots$, and so forth, we can generate a large family of circulant graphs G_{m_1} , G_{m_2} , and so forth. While all of these represent the same original graph G , there is a good reason to regard them as being distinct. This is because each of them has its own k -ft solution, as explained below. Before constructing this solution, however, we need first to convert each graph (G_{m_1} , G_{m_2} , \dots , etc.) into the form defined below.

DEFINITION 3.4. BLOCK GRAPH. For each graph $G_m[a_1, a_2, \dots, a_i : n]$ in the above family, its “block form” is a new circulant graph obtained by multiplying each of its offsets by the inverse m^{-1} . This block graph is denoted $BL(G_m[a_1, a_2, \dots, a_i : n])$.

The reason for converting each graph as defined above follows from the observation proven in [4] which shows that multiplying the offsets of the graph $G_m[a_1, a_2, \dots, a_i : n]$ by m^{-1} will transform each m -distance subset of its offsets into a block of consecutive integers (of the form $\{j, j+1, j+2, \dots\}$). This, in turn, will allow us to compare different k -ft solutions.

The algorithm for constructing a k -ft solution for any given circulant graph G can now be presented as follows. (In this algorithm, the set A denote the offsets of G , and n denote the number of nodes in G .)

ALGORITHM. Fault-tolerance (G, k) .

- (1) Generate all integers $\{m_1, m_2, \dots, m_j\}$, such that for all m_j , we have $\gcd(n, m_j) = 1$ and $1 \leq m_j < (n/2)$.
- (2) For each m_j generated above, find the corresponding partition of the offsets using procedure partition (A, n, m_j) given before. The graph corresponding to this partition is denoted $G_{m_j}[a_1, a_2, \dots, a_i : n]$.
- (3) For each graph $G_{m_j}[a_1, a_2, \dots, a_i : n]$ generated above, construct its corresponding block graph $BL(G_{m_j}[a_1, a_2, \dots, a_i : n])$ as described earlier.
- (4) For each block graph $BL(G_{m_j}[a_1, a_2, \dots, a_i : n])$, use Theorem 2.1. to construct its k -ft solution.
- (5) Compare all k -ft solutions constructed in (4), and select the one with the least node-degree.

It is not difficult to show that the above algorithm is $O(n^2 \log |A| + nk|A|)$ time. For more details on the above algorithm, the reader can refer to reference [4]. Some examples of circulant graphs and their k -ft solutions that are generated by this algorithm are given below.

4. APPLICATIONS TO TWO-DIMENSIONAL MESHES

Next, we examine the application of the formulation presented in the preceding section to the design of fault-tolerant solutions for meshes. We shall only consider meshes of the forms $M[n, n]$

Table 1. Examples of circulant graphs and their 1-ft and 2-ft solutions.

Graph G	1-ft of G	2-ft of G
$G[1,2:10]$	$H[1,2,3:11]$	$H[3,4,5,6:12]$
$G[1,2,4:16]$	$H[4,5,6,7:17]$	$H[4,5,6,7,8:18]$
$G[1,6,11:24]$	$H[5,6,7,8:25]$	$H[5,6,7,8,9:26]$
$G[1,3,5,9:27]$	$H[6,7,8,9,10:28]$	$H[9,10,11,12,13,14:29]$

and $M[n, n, n]$ in this paper. (These are the kinds of meshes we encounter in practice.) We examine first two-dimensional meshes below; and treat three-dimensional meshes in the following section.

DEFINITION 4.1. MULTIDIMENSIONAL MESH. A q -dimensional mesh $M[n_1, n_2, \dots, n_q]$ is a graph whose nodes can be arranged as a q -dimensional matrix, where n_i denotes the number of nodes along the i^{th} dimension. Every node in the mesh is uniquely represented as a q -tuple (x_1, x_2, \dots, x_q) , where x_i is the coordinate of the node along the i^{th} dimension. Two nodes $X = (x_1, x_2, \dots, x_q)$ and $Y = (y_1, y_2, \dots, y_q)$ are connected by an edge if and only if their coordinates differ by one along a single dimension, i.e., there is a single dimension “ i ”, such that $|x_i - y_i| = 1$, and for all other dimensions $j \neq i$, we have $x_j = y_j$. An example of a mesh $M[3, 3]$ was given earlier in Figure 5.

The following proposition develops a 1-ft solution for $M[n, n]$ that is minimal in node degree, i.e., has the same node-degree as the mesh.

PROPOSITION 4.1. The circulant graph $C[1, n : n^2 + 1]$ is a 1-ft extension of $M[n, n]$.

PROOF. Suppose, without loss of generality, that the faulty node in C is the one numbered n^2 (otherwise, renumber the nodes accordingly). Then, a subgraph containing a healthy mesh $M[n, n]$ in C has its rows consisting of the following nodes, numbered in clock-wise direction as, $0, 1, \dots, n-1$ for first row; followed by $n, n+1, \dots, 2n-1$ for second row, and so forth until the last row whose nodes are numbered $n^2 - n, n^2 - n + 1, \dots, n^2 - 1$. The edges joining consecutive nodes in each row correspond to the horizontal edges of the mesh, whereas the edges joining nodes at a distance n in successive rows make the vertical edges of the mesh.

EXAMPLE 4.1. By above results, a 1-ft of $M[3, 3]$ is $C[1, 3 : 10]$ (see again Figure 6).

For values of $k > 1$, the construction of a k -ft will depend on the value of k , and on whether n is even or odd. First, we examine the cases in which n is even.

PROPOSITION 4.2. If n is even, we can construct a k -ft extension of $M[n, n]$ with at most $k + 1$ offsets.

PROOF. To construct the k -ft solution in this case, we first build a 1-ft solution for the mesh $M[n, n]$, (as was shown above in Proposition 4.1.); then, we construct a $(k-1)$ -ft extension of this solution. (Note that the definition of a k -ft is transitive.)

A 1-ft extension of $M[n, n]$, by Proposition 4.1., is a graph $G[1, n : n^2 + 1]$. To construct a $(k-1)$ -ft extension of this graph, first, notice that for any even integer $n > 0$, $n^2 + 1$ and $n-1$ are relatively primes. This is because if any integer D is a divisor of both $n^2 + 1$ and $n-1$, D will also divide the two values $n^2 + 1$ and $(n-1)(n+1)$, and therefore, D must divide the difference $(n^2 + 1) - (n-1)(n+1)$, i.e., D divides 2. Since the original integers $n^2 + 1$ and $n-1$ are both odd, the only possible value for D is 1; i.e., $n^2 + 1$ and $n-1$ must be relatively primes.

Consequently, (by Definition 3.2), we can group the two offsets $\{1, n\}$, of the graph G into an $[n-1]$ -distance subset. Thus, the corresponding block graph H of G has the form $H[1 *$

$(n-1)^{-1}, n * (n-1)^{-1} : n^2 + 1]$, where the multiplication and the inverse operations are done mod $(n^2 + 1)$. It is not difficult to verify that $((n-1) * (n^2 - n)/2) \bmod (n^2 + 1) = 1$, and therefore, $(n-1)^{-1} = (n^2 - n)/2$. Thus, the offsets of H are equal to $\{(n^2 - n)/2, (n^2 - n)/2 + 1\}$, and consequently, a k -ft extension of the mesh is a circulant graph $C[(n^2 - n)/2, (n^2 - n)/2 + 1, \dots, (n^2 - n)/2 + k : n^2 + k]$. ■

EXAMPLE 4.2. By the above proposition, a 2-ft extension of $M[4, 4]$ is $C[6, 7, 8 : 18]$. Similarly, a 4-ft extension of $M[4, 4]$ is $C[6, 7, 8, 9, 10 : 20]$.

In the preceding proposition, if the value of k exceeds $n - 2$, the k -ft solution will reach its center point, i.e., the last offset in the solution will be $\lfloor (n^2 + k)/2 \rfloor$. Beyond this, the cost of a k -ft will grow by only one offset every time the value of k increases by two. This proves the following proposition.

PROPOSITION 4.3. Suppose that n is even and $k \geq n - 1$, then we can construct a k -ft extension of $M[n, n]$ which has at most $\lceil (k + n + 1)/2 \rceil$ offsets. ■

For values of n that are odd, the construction of a k -ft for $M[n, n]$ can be done as shown in the following two propositions.

PROPOSITION 4.4. Suppose that n is odd then we can construct a k -ft extension of $M[n, n]$ which has at most $k + 2$ offsets.

PROOF. Notice first that we can embed a mesh $M[n, n]$ into a circulant graph $C[1, n : n^2]$, where the first row in $M[n, n]$ corresponds to the first n nodes in the graph; the second row correspond to the following n nodes, and so forth. Edges joining consecutive nodes in each row make the horizontal edges of the mesh; whereas edges joining nodes at distance n in successive rows cover the vertical edges.

For any integer $n > 1$, we know that n^2 and $n^2 - 1$ are relatively primes, i.e., $\gcd(n^2, n^2 - 1) = 1$. Moreover, since $n^2 - 1 = (n - 1)(n + 1)$, then $\gcd(n^2, n - 1) = 1$ and $\gcd(n^2, n + 1) = 1$. Thus, we can group the offsets of the graph $G[1, n : n^2]$ into an $[n - 1]$ -distance subset, or $[n + 1]$ -distance subset. (We choose the latter.)

Showing that the offsets $\{1, n\}$ form an $[n + 1]$ -distance subset follows, in a simple way, from Definition 3.2, when we notice that the offsets $\{1, n\}$ are equivalent to $\{-1, n\}$, and the latter two differ by $n + 1$. Therefore, the block graph corresponding to this $[n + 1]$ -partition must have the form $H[1 * (n + 1)^{-1}, n * (n + 1)^{-1} : n^2]$, where the multiplication and the inverse operations are performed mod n^2 . It is easy to check that $(n + 1)^{-1} = (n^2 - n + 1)$, and therefore, the offsets of H , (computed mod n^2), are equal to $\{n^2 - n + 1, n\}$. (Notice that the offsets x , $-x$ and $N - x$ are all the same in any N -node circulant graph.) Thus, by converting the offsets of H into the range 1 to $\lfloor n^2/2 \rfloor$, we get the equivalent offsets $\{n - 1, n\}$, for H ; i.e., H is of the form $H[n - 1, n : n^2]$, and consequently, its k -ft extension is equal to $C[n - 1, n, n + 1, \dots, n + k : n^2 + k]$. Clearly, this solution is also a k -ft extension for the mesh $M[n, n]$. ■

EXAMPLE 4.3. By the above proposition, a 2-ft extension of $M[9, 9]$ is an 83-node circulant graph with four offsets $\{8, 9, 10, 11\}$.

The following proposition demonstrates that for values of $k > n - 2$, a tighter upper-bound on the cost of a k -ft can be proven.

PROPOSITION 4.5. Suppose that n is odd, and $k \geq (n - 3)/2$, then we can construct a circulant graph with at most $\lceil (k + n + 1)/2 \rceil$ offsets that is a k -ft extension of $M[n, n]$.

PROOF. Since n is an odd, therefore n^2 and 2 are relatively primes, i.e., we can form a two-distance partition of the offsets of $G[1, n : n^2]$ and construct its corresponding block graph $H[1 * 2^{-1}, n * 2^{-1} : n^2]$. Notice that $2^{-1} = (n^2 + 1)/2$ in this case, i.e., the offsets of H are $\{(n^2 + 1)/2, (n^3 + n)/2\} = \{(n^2 + 1)/2, (n^2)(n - 1)/2 + (n^2 + n)/2\}$. Since $(n - 1)/2$ is an integer, therefore $(n^2) * (n - 1)/2 \bmod n^2 = 0$, i.e., the offsets of H are $\{(n^2 + 1)/2, (n^2 + n)/2\}$, which when converted to the valid range of offsets will be equal to $\{(n^2 - 1)/2, (n^2 - n)/2\}$.

Therefore, for $k \geq (n-3)/2$, a k -ft extension of H will have the offsets $\{(n^2-n)/2, (n^2-n)/2+1, (n^2-n)/2+2, \dots, (n^2-1)/2 + \lceil k/2 \rceil\}$. Clearly, this solution is also a k -ft extension of $M[n, n]$, and has $\lceil (k+n+1)/2 \rceil$ offsets. ■

5. APPLICATION TO THREE-DIMENSIONAL MESHES

For three-dimensional meshes $M[n, n, n]$, the k -ft solutions will also depend on the values of k and n as explained below. First, when $k = 1$, a solution that is optimal in node-degree can be constructed according to the following proposition.

PROPOSITION 5.1. *The circulant graph $C[1, n, n^2 : n^3 + 1]$ is a 1-ft extension of $M[n, n, n]$.*

PROOF. The proof is a simple generalization of Proposition 4.1.

For $k > 1$, the solution will depend on the value of k , and on whether the value of n is even or odd. We examine first the cases where n is even.

PROPOSITION 5.2. *Suppose that n is even, and $k > 1$, then $M[n, n, n]$ has a k -ft solution that is a circulant graph with at most $2k + 1$ offsets.*

PROOF. The construction of a k -ft solution in this case will be done in two steps; first, we build a 1-ft solution for $M[n, n, n]$ (as was shown before in Proposition 5.1); then, we construct a $(k-1)$ -ft extension of this solution as shown below.

A 1-ft extension of $M[n, n, n]$, by Proposition 5.1, is $G[1, n, n^2 : n^3 + 1]$. To construct a $(k-1)$ -ft extension of G , notice first that for any even integer $n > 0$, $n^3 + 1$ and $n - 1$ are relatively primes. This is because if any integer D is a divisor of both $n^3 + 1$ and $n - 1$, D will also divide the two values $n^3 + 1$ and $(n-1)(n^2 + n + 1)$, and therefore, D must divide the difference $(n^3 + 1) - (n-1)(n^2 + n + 1)$, i.e., D divides 2. Since the original integers $n^3 + 1$ and $n - 1$ are both odd, the only possible value for D is 1. Therefore, $n^3 + 1$ and $n - 1$ must be relatively primes.

Thus, we can form an $[n-1]$ -distance partition of the offsets, and then construct its corresponding block graph $H[1 * (n-1)^{-1}, n * (n-1)^{-1}, n^2 * (n-1)^{-1} : n^3 + 1]$. It is not difficult to verify that for any integer $n > 1$, $(n-1)^{-1} = (n^3 - n^2 - n)/2$, where the inverse operation is computed mod $n^3 + 1$. Thus, the offsets of the block graph H (computed mod $n^3 + 1$) are $\{(n^3 - n^2 - n)/2, (-n^3 - n^2 - n)/2, (-n^3 - n^2 + n)/2\}$. After converting these offsets to the valid range (1 to $\lfloor (n^3 + 1)/2 \rfloor$) we get the following as the offsets of H $\{(n^3 - n^2 - n)/2, (n^3 - n^2 - n)/2 + 1, (n^3 - n^2 + n)/2 + 1\}$. Consequently, a $(k-1)$ -ft solution for H , (which is a k -ft extension of $M[n, n, n]$), has the form, $C[(n^3 - n^2 - n)/2, (n^3 - n^2 - n)/2 + 1, \dots, (n^3 - n^2 - n)/2 + k, (n^3 - n^2 + n)/2 + 1, \dots, (n^3 - n^2 + n)/2 + k : n^3 + k]$. ■

In the preceding proposition, when the value of k exceeds $n - 1$, the gap between the offsets will be completely filled (i.e., we will only have one block of consecutive offsets), where the cost of a k -ft solution in this case will grow by only one offset every time k increases by one. Similarly, if $k > n^2 - n - 2$, the k -ft solution will reach the center point (i.e., the last offset in the solution will be $\lfloor (n^3 + k)/2 \rfloor$), where in this case the cost will grow by only one offset every time k increases by two. These remarks can be used to form the proof of the following proposition.

PROPOSITION 5.3. *Suppose that n is even. Then, if $k > n - 1$, $M[n, n, n]$ has a k -ft solution that is a circulant graph with at most $k + n + 1$ offsets of the form $C[(n^3 - n^2 - n)/2, (n^3 - n^2 - n)/2 + 1, \dots, (n^3 - n^2 + n)/2 + k : n^3 + k]$. Similarly, if $k > n^2 - n - 2$, $M[n, n, n]$ has a k -ft solution that is a circulant graph with at most $\lceil (k + n^2 + n + 1)/2 \rceil$ offsets of the form $C[(n^3 - n^2 - n)/2, (n^3 - n^2 - n)/2 + 1, \dots, \lfloor (n^3 + k)/2 \rfloor : n^3 + k]$. ■*

For odd values of n , several k -ft solutions for $M[n, n, n]$ can be built, where the choice among them will depend on the value of k . But first, we will need the following lemma along the way.

LEMMA 5.1. *For any odd integer x , $x^2 + 1$ is not divisible by 4.*

PROOF. Note first that $x^2 + 1 = (x - 1)(x + 1) + 2 = (x - 1)^2 + 2(x - 1) + 2$. Further, since x is odd, then $(x - 1) \pmod{4} = 0$ or 2 ; and in either case this implies $(x - 1)^2 \pmod{4} = 0$ and $2(x - 1) \pmod{4} = 0$. Thus, $x^2 + 1 \pmod{4} = [(x - 1)^2 + 2(x - 1) + 2] \pmod{4} = [(x - 1)^2 \pmod{4} + 2(x - 1) \pmod{4} + 2 \pmod{4}] = 2$. That is, $x^2 + 1$ cannot be divisible by 4. ■

PROPOSITION 5.4. *Suppose that n is odd, and $k > 1$, then we can construct a k -ft solution for $M[n, n, n]$ that is a circulant graph with at most $2k + 2$ offsets.*

PROOF. The construction of a k -ft solution will be done in two steps; first we build a 1-ft solution for $M[n, n, n]$ (as was shown before in Proposition 5.1); then, we construct a $(k - 1)$ -ft extension of this solution as shown below.

A 1-ft extension of $M[n, n, n]$, by Proposition 5.1, is a graph $G[1, n, n^2 : n^3 + 1]$. To construct a $(k - 1)$ -ft extension of G , we need first to show that $\gcd(n^3 + 1, (n^2 + 1)/2) = 1$. To do this, it suffices to show that $\gcd(n^3 + 1, n^2 + 1) = 2$, since it was proven earlier in Lemma 5.1 that $n^2 + 1$ is not divisible by 4. Thus, let $D = \gcd(n^3 + 1, n^2 + 1)$, where D must be at least 2 (since $n^3 + 1$ and $n^2 + 1$ are both even). This implies that D divides the difference $(n^3 + 1) - (n^2 + 1)$, i.e., D divides $n^2(n - 1)$, which in turn implies that D divides $n^2(n - 1)(n + 1)$, i.e., D divides $n^2(n^2 - 1)$. This means that D divides either n^2 or $n^2 - 1$. However, since D divides $n^2 + 1$, and $D > 1$, therefore, D cannot also divide n^2 . Thus, D must divide both $n^2 + 1$ and $n^2 - 1$, which implies that D divides the difference $(n^2 + 1) - (n^2 - 1)$, i.e., D divides 2. Thus, the only possible value for D is 2.

Thus, we can form an $[(n^2 + 1)/2]$ -distance partition of the offsets of the graph $G[1, n, n^2 : n^3 + 1]$; then construct its corresponding block graph $H[1 * ((n^2 + 1)/2)^{-1}, n * ((n^2 + 1)/2)^{-1}, n^2 * ((n^2 + 1)/2)^{-1} : n^3 + 1]$. It is not difficult to verify that $((n^2 + 1)/2)^{-1} = n^3 - n^2 - n + 2$, where the inverse operation is computed mod $n^3 + 1$. Thus, the offsets of H , (computed mod $n^3 + 1$), are $\{-n^2 - n + 1, -n^2 + n + 1, -n^2 - n - 1\}$. After converting these offsets to the valid range, (by substituting x for every offset $-x$), we will get $H[n^2 + n - 1, n^2 - n - 1, n^2 + n + 1 : n^3 + 1]$. Therefore, a $(k - 1)$ -ft solution for H , (where $k > 1$), is of the form: $C[n^2 - n - 1, \dots, n^2 - n + k - 2, n^2 + n - 1, \dots, n^2 + n + k : n^3 + k]$. ■

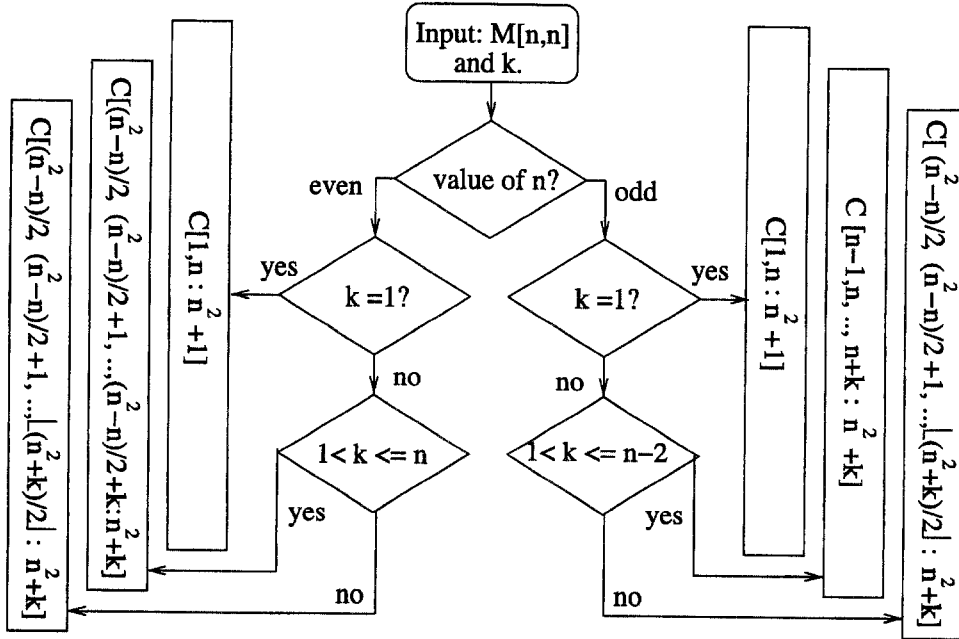
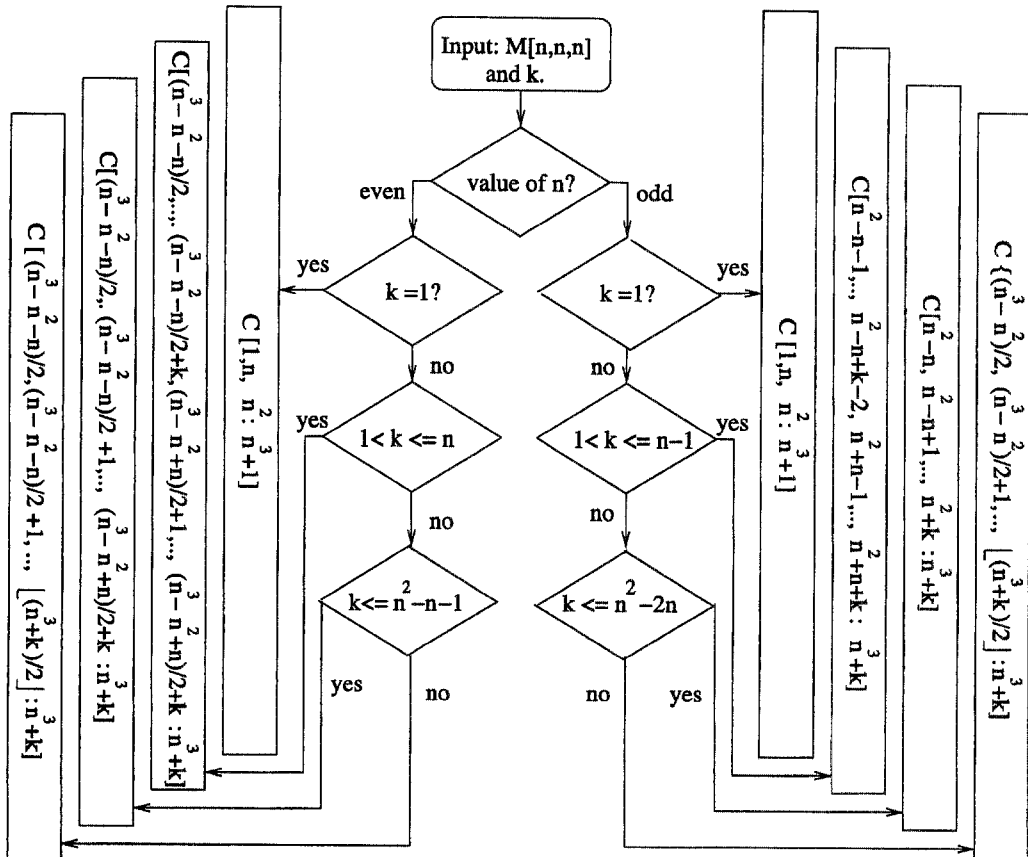
The following proposition demonstrates that for values of $k > n - 1$, a tighter upper-bound on the cost of a k -ft can be proven.

PROPOSITION 5.5. *Suppose that n is odd, and let $k > n - 1$, then $M[n, n, n]$ has a k -ft solution that is a circulant graph with at most $n + k + 1$ offsets.*

PROOF. Before constructing a k -ft solution, we first embed the mesh $M[n, n, n]$ into a circulant graph $G[1, n, n^2 : n^3]$. (This embedding is a simple generalization of that shown earlier in Proposition 4.4.) Then, we construct a k -ft for G as explained below.

Since n^3 and $n^3 + 1$ are relatively primes, and we can factor $n^3 + 1 = (n + 1)(n^2 - n + 1)$, therefore, n^3 and $(n + 1)$ must also be relatively primes. Thus, we can form an $[n + 1]$ -distance partition of the offsets of G ; then construct its corresponding block graph which has the form $H[1 * (n + 1)^{-1}, n * (n + 1)^{-1}, n^2 * (n + 1)^{-1} : n^3]$. By the factorization $n^3 + 1 = (n + 1)(n^2 - n + 1)$, we have $(n + 1)^{-1} = (n^2 - n + 1)$, where the inverse is computed mod n^3 . Thus, by computing the offsets of $H \pmod{n^3}$ we will get the values $\{n^2 - n + 1, -n^2 + n, n^2\}$. By converting the second offset (into the valid range) we shall get the values $\{n^2 - n + 1, n^2 - n, n^2\}$. Thus, if $k > n - 1$, H will have a k -ft solution of the form $C[n^2 - n, n^2 - n + 1, \dots, n^2 + k : n^3 + k]$. ■

The following proposition demonstrates that for values of $k > n^2 - 2n$, a tighter upper bound on the cost of a k -ft can be proven.

Figure 7. Finding a k -ft of $M[n, n]$.Figure 8. Finding a k -ft of $M[n, n, n]$.

PROPOSITION 5.6. Suppose that n is an odd, and let $k \geq (n^2 - n)/2 > 1$, then the mesh $M[n, n, n]$ has a k -ft extension that is a circulant graph with at most $\lceil (n^2 + k + 1)/2 \rceil$ offsets.

PROOF. To construct the solution, we first embed $M[n, n, n]$ into a circulant graph $G[1, n, n^2 : n^3]$ as was explained before; then construct a k -ft solution for G .

To construct a k -ft of G , we can form a two-distance partition of the offsets of G , and then build the block graph H corresponding to this partition, which has the form $H[1 * 2^{-1}, n * 2^{-1}, n^2 * 2^{-1} : n^3]$. Since $2^{-1} = (n^3 + 1)/2$, therefore, it is not difficult to show that the offsets of H , computed mod n^3 , must be equal to $\{(n^3 - n^2)/2, (n^3 - n)/2, (n^3 - 1)/2\}$. Thus, for $k \geq (n^2 - n)/2$, a k -ft extension of H (which is also a k -ft of the mesh $M[n, n, n]$) must have the form $C[(n^3 - n^2)/2, (n^3 - n^2)/2 + 1, \dots, \lfloor (n^3 + k)/2 \rfloor : n^3 + k]$. ■

6. CONCLUSIONS

Many important architectures (such as rings and meshes, for examples) can be modeled as circulant graphs. As a result, these graphs have received a lot of attention and a new method was recently developed for designing fault-tolerant solutions for them [4]. We have reviewed this method in this paper and demonstrated its application to the design of fault-tolerant meshes. Our results for both two and three-dimensional meshes are efficient. These results are summarized in the following two diagrams.

Given a mesh $M[n, n]$, and any value of k , Figure 7 records the cheapest k -ft solution for $M[n, n]$ among those proven before in Section 4. Similarly, given a mesh $M[n, n, n]$ and a value of k , Figure 8 records the cheapest k -ft of $M[n, n, n]$ among those proven earlier in Section 5.

We would like to extend our results to the hypercube configuration. We believe that the approach of offsets partitioning (presented in Section 3) can also be applied to this important architecture. To achieve this, we need first to develop new and efficient ways to embed hypercubes into circulant graphs. We plan to explore this in a future research.

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